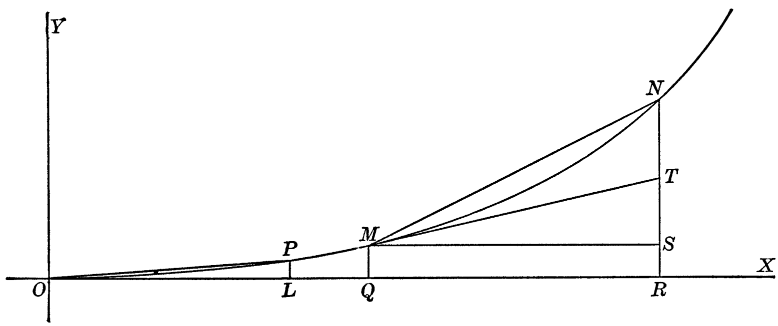
show that, within the limits of ordinary practice, *i. e.*, for angles so small that the difference between arc and sine is inappreciable, the deflection from tangent at m to set n is $(n - m)(n + 2m)$ times the deflection from tangent at 0 to 1.

Solution by W. J. Thome, Detroit, Michigan.

Let the cubic parabola $y = ax^3$ be constructed, and on it let the points named 0, 1, m, and n be represented by O, P, M, and N respectively. Through these points draw the lines parallel to the axes as shown in the figure. Also the chords OP and MN, and MT, the tangent at M.



Since the angles are so small that $\theta = \sin\theta$, we also have $\theta = \tan\theta$ and $\cos\theta = 1$. Also, an arc of the curve, its chord, and the projection of either on the X-axis are all equal. Hence we have

$$\measuredangle LOP = \tan LOP = \frac{LP}{OL} = \frac{a(OL)^3}{OL} = a(OL)^2 = a(OP)^2 = a(1)^2 = a,$$

$$\measuredangle SMN = \tan SMN = \frac{SN}{MS} = \frac{RN - RS}{QR} = \frac{RN - QM}{OR - OQ} = \frac{a(OR)^3 - a(OQ)^3}{OR - OQ} = \frac{a(ON)^3 - a(OM)^3}{ON - OM}$$

$$= \frac{a(n)^3 - a(m)^3}{n - m} = \frac{a(n^3 - m^3)}{n - m} = a(n^2 + nm + m^2),$$

$$\measuredangle SMT = \tan SMT = \frac{dy}{dx} = 3ax^2 = 3a(OQ)^2 = 3a(OM)^2 = 3am^2,$$

$$\begin{aligned}\measuredangle TMN &= \measuredangle SMN - \measuredangle SMT\\ &= a(n^2 + nm + m^2) - 3am^2 = a(n^2 + nm - 2m^2)\\ &= (n - m)(n + 2m)a\\ &= (n - m)(n + 2m)\measuredangle LOP.\end{aligned}$$

525 (Geometry). Proposed by C. N. SCHMALL, New York City.

Given a quadrant of a circle AOB, where OA and OB are bounding radii, and a semicircle ACO having OA as a diameter and lying on the same side as the quadrant. Describe a circle which shall touch the two arcs and the radius OB.

I. Solution by W. J. Thome, Detroit, Mich.

Suppose the problem solved and the figure constructed.

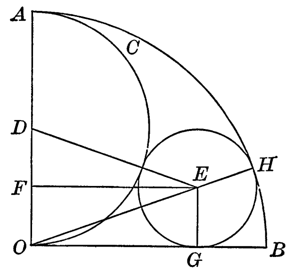
Let O and D be the centers of the two given circles and let E be the center of the required circle. Draw DE. Draw OE and extend it until it intersects the given quadrant at H. Through E draw EF and EG perpendicular to OA and to OB, respectively.

The problem may be considered solved if we can obtain a value of EG, the radius of the required circle. Let $OA = 2R$, $OD = R$, and $EG = r$. Then $OE = 2R - r$, $DE = R + r$, and $DF = R - r$. Now $\overline{OE}^2 - \overline{EG}^2 = \overline{DE}^2 - \overline{DF}^2$, since $OG^2 = FE^2$, or

Whence,

$$(2R - r)^2 - r^2 = (R + r)^2 - (R - r)^2.$$

$$r = \frac{R}{2}, \quad \text{or} \quad EG = \tfrac{1}{2}OD.$$



II. Solution by Horace Olson, Chicago, Illinois.

Let x and y be the coördinates of the center of the required circle, referred to OB and OA as the positive directions on the x and y axes, respectively. Then by the conditions of the problem:

$$y = r - \sqrt{x^2 + y^2}, \qquad y = \sqrt{\left(\frac{r}{2} - y\right)^2 + x^2} - \frac{r}{2}; \qquad r = OA;$$

whence

$$r^2 - 2ry - x^2 = 0, \quad \text{and} \quad 2ry - x^2 = 0.$$

Subtracting, we have $r^2 - 4ry = 0$; whence $y = \frac{1}{4}r$ and $x = \pm\frac{1}{2}\sqrt{2}r$. The value $x = +(r/2)\sqrt{2}$ is evidently the only one applying to this problem. The radius of the required circle is therefore $\frac{1}{4}r$.

A solution similar to the second above was received from Elijah Swift.

435 (Calculus). Proposed by B. F. FINKEL, Drury College.

Show that

$$\int_0^\infty e^{-x^2-(a^2/x^2)}dx = \frac{\sqrt{\pi}}{2e^{2a}}$$

by a transformation rather than by the usual method of differentiating under the sign of integration, as, for example, in Byerly's *Integral Calculus*, pages 106–107.

Note by William Hoover, Columbus, Ohio.

On this interesting equation, reference may be made to an article by William Walton in the *Quarterly Journal of Mathematics*, Vol. XII, p. 181, On the Evaluation of a Pair of Definite Integrals, a few lines opening it being as follows:

"The evaluation of the two definite integrals

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]\cos\alpha} \cdot \cos\left\{\left(x^2 + \frac{c^2}{x^2}\right)\sin\alpha\right\}dx,$$

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]\cos\alpha} \cdot \sin\left\{\left(x^2 + \frac{c^2}{x^2}\right)\sin\alpha\right\}dx$$

is effected in textbooks on the integral calculus by the substitution of an impossible expression for k in the known relation

$$\int_0^\infty e^{-[x^2+(c^2/x^2)]k}dx = \frac{\pi^{1/2}}{2k^{1/2}} \cdot e^{-2ck}.$$

"As such a method of arriving at the evaluation of definite integrals is regarded by eminent writers on the subject as suggestive rather than demonstrative, I think that the following evaluations of the two integrals in question may be of some interest to students."